

hep-th/0108095  
TIFR/TH/01-29

# Effective Action for D-branes on $SU(2)/U(1)$ Gauged WZW Model

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## Abstract

Dynamics of D-branes on  $SU(2)/U(1)$  gauged WZW model are investigated. We find the effective action for infinite  $k$ , where  $k$  is the level of WZW model. We also consider finite  $k$  correction to the effective action which is compatible with Fedosov's deformation quantization of the background.

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# 1 Introduction

Initially, D-branes in string theory were defined as hyper-surfaces on which open strings can end [1]. More precisely, the branes can be considered as boundary states in the close string picture, subject to some gluing condition [2]. The earlier works of Ishibashi [3] and Cardy [4] for RCFT, make it possible to construct the boundary states for those backgrounds on which the representation of string states are known.

On the other hand, it is known that there is a non-commutative structure on the branes in a flat background [5, 6]. This non-commutativity can be formulated by a  $*$  product known as Moyal product with a constant non-commutativity parameter  $\theta$ . Again on a curved manifold, the Moyal product is ill-defined and the  $\theta$ , effectively, is non-constant.

So the important task is, firstly, finding Cardy's solutions for a given background, secondly, looking for an effective action for which a consistent non-commutative structure is expected. For the flat space, the result is a non-commutative gauge theory (or a matrix model for D0-branes) with Moyal brackets. But, for a general background, it is still an open problem.

To study this subject, important special classes are group manifolds and quotient spaces. The D-branes on group manifolds were introduced in [7] and their effective action and non-commutative structure were studied in [8]. Recently, the quotient space,  $SU(2)/U(1)$  in the frame of gauged WZW models has been studied in [9]. This space topologically is a disk with radius  $\sqrt{k}$ , where  $k$  is the level of WZW model. Several D-branes are introduced on this disk and it is the purpose of the present paper to study the dynamics of these branes.

This paper is organized as follows. In the second section, we review the introduction of the space and its branes. In section 3, a D2-brane is studied by DBI action. We find the mass of D2-brane and the open string spectrum as fluctuating modes on the brane. In the fourth section, we compute the effective action from boundary conformal field theory. The result is a non-commutative  $U(1)$  gauge theory. This result is valid only for  $k \rightarrow \infty$ , but in section 5, we extend the effective action for finite  $k$  corrections and show that the result is some non-commutative theory with a non-constant  $\theta$ . We show that this non-commutativity is consistent with the known Fedosov's  $\star$  product on the disk. The Fedosov's formalism is a deformation quantization on a curved space which gives a natural definition for the non-commutativity on the curved space.

## 2 D-Branes on $SU(2)/U(1)$ WZW

The  $SU_k(2)$  WZW model can be considered as a sigma model on  $S^3$  with an antisymmetric B-field. Firstly, consider the  $S^3$  metric as follows,

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\tilde{\phi}^2 \quad (2.1)$$

for which one can use the Euler's angels parameterization of a group element,  $g \in SU(2)$ ,

$$g = e^{i\chi\sigma^3/2} e^{i\tilde{\theta}\sigma^1/2} e^{i\varphi\sigma^3/2} \quad (2.2)$$

where,

$$\chi = \tilde{\phi} + \phi, \quad \varphi = \tilde{\phi} - \phi, \quad \tilde{\theta} = 2\theta. \quad (2.3)$$

The anti-symmetric B-field in the Wess-Zumino term of WZW model is as follows,

$$B = \sin^2 \theta \, d\phi \wedge d\tilde{\phi} \quad (2.4)$$

On this background, the WZW action will be,

$$\begin{aligned} S_{SU(2)} &= k \int d^2 z (\partial\theta\bar{\partial}\theta + \sin^2 \theta \partial\phi\bar{\partial}\phi + \cos^2 \theta \partial\tilde{\phi}\bar{\partial}\tilde{\phi} + \sin^2 \theta (\partial\phi\bar{\partial}\tilde{\phi} - \bar{\partial}\phi\partial\tilde{\phi})), \quad (2.5) \\ &= k \int d^2 z (\partial\theta\bar{\partial}\theta + \tan^2 \theta \partial\phi\bar{\partial}\phi + \cos^2 \theta (\partial\tilde{\phi} + \tan^2 \theta \partial\phi)(\bar{\partial}\tilde{\phi} - \tan^2 \theta \bar{\partial}\phi)) \quad (2.6) \end{aligned}$$

Now we consider the gauged model,  $SU(2)/U(1)$ , by gauging a  $U(1)$  corresponding to shifting  $\tilde{\phi}$ . It needs introducing of gauge fields in the action as,

$$S = k \int d^2 z (\partial\theta\bar{\partial}\theta + \tan^2 \theta \partial\phi\bar{\partial}\phi + \cos^2 \theta (\partial\tilde{\phi} + \tan^2 \theta \partial\phi + A_z)(\bar{\partial}\tilde{\phi} - \tan^2 \theta \bar{\partial}\phi + A_{\bar{z}})) \quad (2.7)$$

Then by integrating out  $A$  fields, one finds a sigma model with the following metric and non-constant dilaton [9]:

$$ds^2 = \frac{k}{1-r^2} (dr^2 + r^2 d\phi^2) \quad (2.8)$$

$$g_s(r) \equiv e^\Phi = g_s(0)(1-r^2)^{-\frac{1}{2}}. \quad (2.9)$$

where  $r = \sin \theta$ . This is a disk with boundary at  $r = 1$ . Although, it seems that the target space has  $U(1)$  symmetry corresponding to the shift of  $\phi$ , actually it is broken to a  $\mathbf{Z}_k$  symmetry. This can be seen explicitly by finding the divergence of the current of  $\phi$  shift,  $\partial_\alpha j^\alpha \sim k F_{z\bar{z}}$  where  $F$  is the field strength of  $A$  field. Now a  $\mathbf{Z}_k$  subgroup of  $U(1)$  is non-anomalous and is a symmetry of the model.

On the other hands, from the GKO construction, the  $SU_k(2)/U_k(1)$  is known as *parafermion* model. This model has  $\mathbf{Z}_k \times \overline{\mathbf{Z}}_k$  symmetry. A representation in this model can be shown by  $(j, n)$  where  $j = 1, \frac{1}{2}, 1, \dots, \frac{k}{2}$  is the  $SU(2)$  spin and  $n = -k+1, -k+2, \dots, k$ . We need also the identification  $2j+n=0 \pmod{2}$ .

In the parafermion language, the D-branes are introduced as boundary states. Consider  $(J \pm \bar{J})|Boundary\rangle = 0$  as gluing conditions with corresponding solutions as  $A$ -states and  $B$ -states, respectively. The  $A$ -solutions can be constructed as Ishibashi states with identical left and right sectors,

$$|A; jn\rangle\rangle = \sum_{States} |j, n\rangle \otimes |\widetilde{j, n}\rangle \quad (2.10)$$

Then the following linear combination gives Cardy states with correct modular transformations [4],

$$|A; \hat{j}\hat{n}\rangle_C = \sum_{j,n \in PF} \frac{S_{\hat{j}\hat{n}}^{PF \, jn}}{\sqrt{S_{00}^{PF \, jn}}} |A; jn\rangle\rangle \quad (2.11)$$

where  $S^{PF}$  is the modular transformation matrix of parafermionic theory and is introduced as follows [9],

$$S_{j\hat{n}}^{PF\ jn} = 2\sqrt{\frac{1}{k(k+2)}} e^{\frac{i\pi n\hat{n}}{k}} \sin\left(\frac{\pi(2\hat{j}+1)(2j+1)}{k+2}\right). \quad (2.12)$$

These A-branes have geometric interpretations as D0-branes on some special points on the disk circumstance and also D1-branes as lines between these special points. Also it is possible to find the B-branes through T-duality transformation on the disk.<sup>†</sup> The results are D2-branes co-centered with the disk and D0-branes near the center of the disk.<sup>‡</sup>

### 3 DBI Action for D2-Brane with Flux Stabilization

In  $SU(2)$  model, D2-branes are two-spheres and for  $j \ll k$ , they can be considered as bound states of  $(2j+1)$  D0-branes. These D2-branes go to flattened spheres or equivalently, two sided disks in  $SU(2)/U(1)$  theory. To support this idea, we follow the procedure of [9] and [11] to find the open string spectrum on this D2-brane.

Consider a D2-brane as a disk with radius  $r_m$  centered at  $r = 0$ . This brane can be stabilized by an  $F$  field with fixed flux equal to the number D0-branes on it. So for a two sided D2-brane, we have,<sup>§</sup>

$$\frac{N}{k} = \frac{(2\hat{j}+1)}{k} = 2\frac{1}{2\pi} \int_{D2} \frac{F}{k} = \frac{1}{\pi} \int_0^{r_m} dr f(r), \quad (3.1)$$

in which  $N = (2\hat{j}+1)$  is the number of D0-branes near the center and  $f(r) := 2\pi F_{r\phi}/k$ .

Now, we can minimize the DBI action for the above D2-brane in the given background (2.8) and (2.9). Consider  $\lambda$  as a Lagrange multiplier to impose the fixed flux condition (3.1), one can minimize the following quantity:

$$\frac{k}{2\pi g_s(0)} \int d\phi dr \sqrt{1-r^2} \sqrt{r^2/(1-r^2)^2 + f^2} - k\lambda \left( \frac{1}{\pi} \int_0^{r_m} f(r') dr' - \frac{N}{k} \right), \quad (3.2)$$

and the corresponding  $f$  will be [9],

$$f_m = \frac{r}{(1-r^2)} \frac{\sqrt{1-r_m^2}}{\sqrt{r_m^2-r^2}}. \quad (3.3)$$

This field strength can be derived from the following gauge field,

$$A_\phi = \frac{k}{2\pi} \tan^{-1} \frac{\sqrt{r_m^2-r^2}}{\sqrt{1-r_m^2}}, \quad (3.4)$$

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<sup>†</sup>This T-duality interchanges the role of circumstance and center of the disk with the map,  $r' = \sqrt{1-r^2}$ , see [9].

<sup>‡</sup>Besides these branes, there are other branes on the disk, but for our purpose, we consider only the above mentioned B-branes (see [9]).

<sup>§</sup>Take units with  $\alpha' = 1$ .

and  $r_m$  can be found in terms of the flux [9],

$$\frac{\pi(2\hat{j}+1)}{k} = \int_0^{r_m} dr f(r) = \tan^{-1} \frac{r_m}{\sqrt{1-r_m^2}}, \quad (3.5)$$

or

$$r_m = \sin \left( \frac{\pi(2\hat{j}+1)}{k} \right). \quad (3.6)$$

In [11], the mass of D2-brane in  $SU(2)$  background is computed as the minimum of the DBI action. Also by considering small fluctuations around the classical solution, the open string spectrum has been found [11]. Now, we follow [11] to find the D-brane mass and open string spectrum for  $SU(2)/U(1)$  case. Firstly, put back  $f_m$  in the DBI action, we will find the mass of D2-brane as follows,

$$M_{D2} = S|_{f_m} = \frac{k}{g_s(0)} r_m = \frac{k}{g_s} \sin \left( \frac{\pi(2\hat{j}+1)}{k} \right). \quad (3.7)$$

This mass is comparable with that derived from the overlap of D-brane state with  $|A; j = 0, n = 0 \gg$  state, for large  $k$  (see eq. (3.8) in [9]).

To find open string spectrum on this D2-brane, we turn on small fluctuations around the classical solution as follows,

$$\tilde{A}_\phi = \frac{k}{2\pi} \tan^{-1} \frac{\sqrt{r_m^2 - r^2}}{\sqrt{1-r_m^2}} + \frac{k}{2\pi} a_\phi \quad (3.8)$$

$$\tilde{A}_r = \frac{k}{2\pi} a_r, \quad A_t = 0 = \tilde{A}_t \quad (3.9)$$

In terms of these fields, we can write  $G + 2\pi F$  matrix as,

$$G + 2\pi F = k \begin{pmatrix} \frac{-1}{k} & \partial_t a_r & \partial_t a_\phi \\ -\partial_t a_r & \frac{1}{1-r^2} & \partial_r \tan^{-1} \frac{\sqrt{r_m^2 - r^2}}{\sqrt{1-r_m^2}} + f_{r\phi} \\ -\partial_t a_\phi & -\partial_r \tan^{-1} \frac{\sqrt{r_m^2 - r^2}}{\sqrt{1-r_m^2}} - f_{r\phi} & \frac{r^2}{1-r^2} \end{pmatrix} \quad (3.10)$$

where  $f_{r\phi} := \partial_r a_\phi - \partial_\phi a_r$ . Inserting these into DBI action and expanding up to second order, we find the following equation of motion for fluctuating fields,

$$\ddot{f}_{r\phi} = \frac{1}{k} \left( \partial_r \left( \frac{r(1-r^2)}{\sqrt{r_m^2 - r^2}} \partial_r \frac{(r_m^2 - r^2)^{3/2}}{r(1-r^2)} \right) + \frac{r_m^2 - r^2}{r^2} \partial_\phi^2 \right) f_{r\phi} \quad (3.11)$$

For  $r_m \rightarrow 1$  and with  $r = \sin \theta$ , it will be,

$$\ddot{\tilde{f}} = \frac{1}{k} \left( \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial_\phi^2 - \partial_\phi^2 \right) \tilde{f} \quad (3.12)$$

where  $\tilde{f} := f_{r\phi} \cot \theta$ . The above equation can be written in terms of the Laplace's operator of  $S^2$  and  $J_3^2$ , the third component of angular momentum operator, *i.e.*,

$$\ddot{\tilde{f}} = -\frac{1}{k} (\square_{S^2} - J_3^2) \tilde{f} \quad (3.13)$$

The eigenvalues are well-known,

$$M^2 = \frac{j(j+1)}{k} - \frac{m^2}{k}. \quad (3.14)$$

This is in agreement with conformal dimensions of the GKO construction of  $SU(2)/U(1)$ , for large  $k$ .

## 4 Effective Action from Boundary CFT

In this section, we use boundary CFT formalism to find the effective action of D0-branes. Formally, it is possible to find the effective action term by term from the  $n$ -point functions of vertex operators on the brane. For definiteness, we consider  $J^\mu V[A_\mu] : (x_1)$  as boundary field in which  $J^\mu$  is the current and  $V[A_\mu] = \sum_\alpha a_\mu^\alpha V[\alpha]$  is a lowest weight vertex operator in some suitable basis. Now, the cubic interaction in the effective action will corresponds to the following 3-point function,<sup>¶</sup>

$$< : J^\mu V[A_\mu] : (x_1) : J^\nu V[A_\nu] : (x_2) : J^\rho V[A_\rho] : (x_3) > \quad (4.1)$$

To calculate the above 3-point function, three kinds of OPE are needed. Firstly, the OPE of currents  $J$ 's or currents algebra. Secondly, the OPE of currents and vertex operators which is the definition of current algebra primaries, and finally, the OPE of vertex operators which corresponds to the fusion algebra.

Our case,  $SU_k(2)/U_k(1)$  or parafermionic theory, has  $\mathbf{Z}_k \times \overline{\mathbf{Z}}_k$  symmetry for left and right handed sectors. To realized it, it is common to introduce currents [10],  $\psi_l(z)$  and  $\bar{\psi}_l(\bar{z})$  (for  $l = 1, 2, \dots, k$ ), with  $(l, 0)$  and  $(0, l)$  charges under  $\mathbf{Z}_k \times \overline{\mathbf{Z}}_k$  symmetry, respectively. These currents have also conjugate partners, but they are related as follows,

$$\begin{aligned} \psi_l^\dagger(z) &= \psi_{k-l}(z) \\ \bar{\psi}_l^\dagger(\bar{z}) &= \bar{\psi}_{k-l}(\bar{z}) \end{aligned} \quad (4.2)$$

The currents algebra for the left handed currents is,

$$\begin{aligned} \psi_l(z)\psi_{l'}(w) &= c_{ll'}(z-w)^{-2ll'/k} \psi_{l+l'}(w) + \dots & l+l' < k \\ \psi_l(z)\psi_{l'}^\dagger(w) &= c_{ll'}(z-w)^{-2l(k-l)/k} \psi_{l-l'}(w) + \dots & l < l' \\ \psi_l(z)\psi_l^\dagger(w) &= (z-w)^{-2l(k-l)/k} \left( I + \frac{2d_l}{c}(z-w)^2 T(w) + \dots \right) \end{aligned} \quad (4.3)$$

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<sup>¶</sup> $\mu, \nu, \rho = 1, 2, 3$  for  $SU(2)$  case. For  $SU(2)/U(1)$  branes, with large  $k$  limit, indices are  $a, b, c = 1, 2$ , see below.

where  $c_{ll'}$  is a constant,  $I$  the identity operator,  $T(z)$  the energy-momentum tensor,  $d_l$  the conformal weights of  $\psi_l$  (the same as  $\psi_l^\dagger$ ) and  $c$  the central charge. The two latter's are given as,

$$d_l = \frac{l(k-l)}{k}, \quad \text{and} \quad c = \frac{2(k-1)}{k+2}. \quad (4.4)$$

The relation between the parafermions and the original  $SU(2)$  currents can be seen from the following realization of  $SU(2)$  algebra [10],

$$\begin{aligned} J_+ &= \frac{\sqrt{k}}{\sqrt{2}} \psi_1 e^{i\frac{\sqrt{2}}{\sqrt{k}}h} \\ J_- &= \frac{\sqrt{k}}{\sqrt{2}} \psi_1^\dagger e^{-i\frac{\sqrt{2}}{\sqrt{k}}h} \\ J_3 &= \frac{\sqrt{k}}{\sqrt{2}} i\partial h \end{aligned} \quad (4.5)$$

The  $SU(2)/U(1)$  is reached by gauging  $J_3$  and it roughly means to drop  $h$  in (4.5). The remaining currents  $\psi_1$  and  $\psi_1^\dagger$  have not a closed algebra, instead they generate the  $2k$ -dimensional algebra (4.3), for finite  $k$ .

Now, let  $V[A_a] = \sum_{j,n} a_a^{jn} V[jn]$  be an arbitrary linear combination of vertex operators,  $V[jn]$ . Then the OPE of currents and the primaries is as follows [10],

$$\psi_l(z)V[A_a](w) = \frac{1}{(z-w)^{d_l}} V[T_l A_a](w), \quad (4.6)$$

where  $T_l$  is the representation of  $\psi_l$  in the given basis. The OPE of these vertex operators can be written as,

$$V[jn](z)V[j'n'](w) = \sum_{j''n''} (z-w)^{\Delta_{j''n''}-\Delta_{jn}-\Delta_{j'n'}} V[j''n''](w), \quad (4.7)$$

in which  $\Delta_{jn} = \frac{j(j+1)}{k+2} - \frac{n^2}{4k}$  is the conformal weight of primaries and  $[j''n'']$ 's are all representations in the fusion  $[jn] * [j'n']$ .

Since we are looking for the low energy effective action, it is sufficient to look at the decoupling limit [6] which corresponds to massless states and can be reached by  $k \rightarrow \infty$ . In this limit, it can be seen from (4.3) that  $\psi_1$  and  $\psi_1^\dagger$  have a closed algebra in the following form:

$$\psi_1(z)\psi_1^\dagger(w) = \frac{I}{(z-w)^2} \quad (4.8)$$

It is convenient to introduce  $j_1 = (\psi_1 + \psi_1^\dagger)/\sqrt{2}$  and  $j_2 = i(\psi_1 - \psi_1^\dagger)/\sqrt{2}$ , then we have,

$$j_a(z)j_b(w) = \frac{\delta_{ab}}{(z-w)^2} \quad (4.9)$$

$$j_a(z)V[A_b](w) = \frac{1}{(z-w)} V[L_a A_b](w), \quad (4.10)$$

where  $a, b = 1, 2$  and  $L_a$  is the representation of  $j_a$ .

Also the weights of vertex operators are zero for  $k \rightarrow \infty$ , so from (4.7),

$$V[A_a](z)V[A_b](w) = V[A_a * A_b](w), \quad (4.11)$$

where we have used  $*$  as a notation for fusions of the representations. However, this  $*$  product indeed defines a non-commutative product. On the flat space the  $*$  product comes from the non-commutativity of coordinates on the boundary. For  $SU(2)$  case, it can be found directly from the fusion algebra of  $SU_k(2)$  (see [8]). For the parafermionic theory, it can be understood as a reduced form of  $*$  product in  $SU_k(2)$  theory. However, we postpone the definition to section 5.

With (4.9) to (4.11), the OPE's are similar to a free theory in flat space [8, 6] and it is easy to find the 3-point function (4.1). Also a similar calculation can be done for 4-point function and the results are as follow, (see [8] for details)

$$S_{(3)} \sim L_a A_b * [A^a, A^b]_* \quad (4.12)$$

$$S_{(4)} \sim [A_a, A_b]_* * [A^a, A^b]_* \quad (4.13)$$

The quadratic or mass term of the action can be read from the mass operator,

$$M^2 = \frac{J_\mu J^\mu}{k+2} - \frac{(J_3)^2}{k} \quad (4.14)$$

In large  $k$  limit it will be,

$$M^2 = \frac{1}{k} (J_\mu J^\mu - (J_3)^2) = \frac{1}{k} (J_+ J_- + J_- J_+) \quad (4.15)$$

The last expression is equivalent to  $\psi_1 \psi_1^\dagger \sim j_a j^a \sim L_a L^a$ , so we will have the following quadratic term in the effective action,

$$S_{(2)} \sim L_a A_b * L_a A_b \quad (4.16)$$

Now one can arrange the above terms in the following action,

$$S = \frac{1}{4} \text{Tr}(F_{ab} * F^{ab}) \quad (4.17)$$

where  $F_{ab} = L_a A_b - L_b A_a + [A^a, A^b]_*$ . This result can be obtained by reduction from  $SU(2)$  case as has been done in [12].

Two remarks on this effective action are worth mentioning:

- The action (4.17) is equivalent to that on a plane and there is no signature for the disk. Indeed, by sending  $k \rightarrow \infty$ , the disk goes to the entire plane. One can rescale coordinates on the disk as  $r \rightarrow \tilde{r}/\sqrt{k}$ , then the metric will be,

$$ds^2 = \frac{k}{k - \tilde{r}^2} (d\tilde{r}^2 + \tilde{r}^2 d\phi^2) \quad (4.18)$$

In limit  $k \rightarrow \infty$  with fixed  $\tilde{r}$ , this will be a flat plane.

- The  $*$  product has not yet been identified with a geometric construction. On the flat space the  $*$  product in the action comes from the non-commutativity of coordinates on the boundary. For non-flat backgrounds such as the disk (4.18), with finite radius, we discuss a possible definition in section 5.



## 5 Finite $k$ Corrections to the Effective Action and Fedosov's $\star$ -product

As mentioned in the end of the previous section, for  $k \rightarrow \infty$  limit, the disk is equivalent to a plane. On the other hands, a geometric interpretation needs large  $k$  limit, thus, to find a finite radius disk, we have to consider  $\frac{1}{k}$  corrections.

Such corrections with a signal for a disk solution was found in [13] in a different context. In [13], the static action is suggested to be,

$$S = \int \text{Tr} \left( -\frac{1}{4} [X_a, X_b]^2 - \frac{1}{k} (X_a X^a) \right). \quad (5.1)$$

where  $X_a$  stands for  $A_a$  in the previous section and can be understood as coordinates of the background geometry. The second term in (5.1) comes from a constant dilaton background or can be considered as the leading order term of dilaton in  $\sqrt{1-r^2}$ .

The equation of motion will be as follows,

$$[X^a, [X_a, X_b]] - \frac{2}{k} X_b = 0. \quad (5.2)$$

There are two interesting classes of solutions. Firstly, consider a 3-dimensional action ( $a, b = 1, 2, 3$ ). It has the following solution[13],

$$X_a = \frac{1}{2\sqrt{k}} \tau_a, \quad (5.3)$$

where  $\tau_a$ 's are some irreducible representation of the  $SU(2)$  Pauli matrices. So we have,

$$X_1^2 + X_2^2 + X_3^2 = R^2 \quad (5.4)$$

in which  $R$  is a constant depending on  $k$  and the dimension of irreducible representation. The solution (5.3) with (5.4) is a 2-sphere.

Another class of solution to (5.2) is the following,

$$X_1 = \frac{1}{\sqrt{2k}} \tau_1, \quad X_2 = \frac{1}{\sqrt{2k}} \tau_2, \quad X_3 = 0. \quad (5.5)$$

This solution can be interpreted as a disk which is a squeezed version of the sphere in (5.4).

Now we extend the action (5.1) by adding next to leading order in  $1/k$  expansion of dilaton potential  $\sqrt{1-r^2/k} = 1 - r^2/2k + \dots$ ,

$$S = \int \text{Tr} \left( -[X_1, X_2]^2 - \frac{1}{k} (X_1^2 + X_2^2) + \frac{1}{2k^2} (X_1^2 + X_2^2)^2 \right). \quad (5.6)$$

with the following equation of motion,

$$[X_1, [X_1, X_2]] - \frac{2}{k} X_2 \left( 1 - \frac{r^2}{k} \right) = 0. \quad (5.7)$$

In (5.6) and (5.7), we have replaced  $X_a \rightarrow X_a/\sqrt{k}$  to have correct large  $k$  limit. The equation of motion (5.7) has a solution as,

$$[X_1, X_2] = -i\left(1 - \frac{r^2}{k}\right) \quad (5.8)$$

To interpret this solution as a disk, we need to define the products in the above relations correctly. Indeed, (5.8) can define a  $\star$  product as follows,

$$X_1 \star X_2 - X_2 \star X_1 = -i\left(1 - \frac{X_1^2 + X_2^2}{k}\right). \quad (5.9)$$

In  $k \rightarrow \infty$  limit, the commutator (5.8) is  $[X_1, X_2] = -i$  which is a plane brane solution in flat space.

It can be shown that the  $\star$  product (5.9) is consistent with a natural non-commutative structure on the given disk in (2.8). This non-commutative structure is obtained from a deformation quantization procedure known as Fedosov's formalism. In this formalism, the basic idea is to find a map between the space of functions on a symplectic curved manifold to its Weyl algebra bundle on which the ordinary Moyal product can be defined. By inverting the map, one can find the product rule for functions on the manifold.

Here, we will briefly give the basic prescription based on [15] for circularly symmetric two dimensional spaces. Consider a symplectic manifold  $M$  with  $\Omega_0$  as its symplectic form. Let  $W$  be the Weyl algebra bundle with an Abelian connection  $D$  and an  $\circ$  product as the ordinary Moyal product:

$$\circ := \exp\left(\frac{i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial y^i} \omega^{ij} \frac{\overrightarrow{\partial}}{\partial y^j}\right) \quad (5.10)$$

where  $\omega^{ij}$  is a constant antisymmetric parameter and  $\hbar$  is a deformation parameter. Now for  $W_D \equiv \ker D \subset W$ , we can define an invertible map from  $C^\infty(M)[[\hbar]]$  to  $W_D$ ,

$$Q : C^\infty(M)[[\hbar]] \rightarrow W_D \quad (5.11)$$

with inverse:

$$\sigma : W_D \rightarrow C^\infty(M)[[\hbar]] \quad (5.12)$$

Then Fedosov's  $\star$  product on  $C^\infty(M)[[\hbar]]$  can be defined by,

$$a_0 \star b_0 := \sigma(Q(a_0) \circ Q(b_0)), \quad a_0, b_0 \in C^\infty(M)[[\hbar]] \quad (5.13)$$

This product is associative and is a deformation of the Poisson bracket with the symplectic form  $\Omega_0$ .

Now for a two dimensional circularly symmetric space,

$$ds^2 = \Lambda(r)(dr^2 + r^2 d\phi^2), \quad (5.14)$$

the map  $Q$  can be defined as follows [15],

$$\begin{aligned} a = Q(a_0(r, \phi)) &= a_0\left(G(r, y_1), \phi + \frac{y_2}{r}\right) \\ a \in W_D, \quad a_0 &\in C^\infty(M)[[\hbar]] \end{aligned} \quad (5.15)$$

where  $G(r, y_1)$  is found in,

$$\int_r^{G(r, y_1)} \Lambda(r') dr' = y_1 r \quad (5.16)$$

With  $Q$  in hand, one can define the  $\star$  product on  $M$  by (5.13),

$$a(r, \phi) \star b(r, \phi) = \left( a_0 \left( G(r, y_1), \phi + \frac{y_2}{r} \right) \circ b_0 \left( G(r, y_1), \phi + \frac{y_2}{r} \right) \right)_{y_1=y_2=0} \quad (5.17)$$

Now consider the special case of the disk metric in (4.18) for which  $\Lambda(r) = k/(k - r^2)$  and  $G(r, y_1)$  can be found as,

$$G(r, y_1) = \sqrt{k - (k - r^2)e^{-2y_1 r/k}} \quad (5.18)$$

It is simple to compute the  $\star$  product for  $X_1 = r \cos \phi$  and  $X_2 = r \sin \phi$  as functions on  $M$ ,

$$X_1 \star X_2 - X_2 \star X_1 = -i\hbar \left( 1 - \frac{r^2}{k} \right). \quad (5.19)$$

It is just equivalent to the relation (5.8) (with  $\hbar = 1$ ). This means that Fedosov's formalism on the disk confirms the action (5.6) with the solution (5.8).

## 6 Conclusion

The effective action for branes on  $SU_k(2)/U_k(1)$  gauged WZW is found to be a non-commutative gauge theory on a disk, for infinite  $k$  limit. This effective action is equivalent to that for a plane, and this is not surprising, since the disk is a plane for  $k \rightarrow \infty$ .

On the other hand, finite  $k$  corrections are possible, which correspond to new terms in the matrix model/DBI, respecting non-constant dilaton background. The resulting action with  $1/k$  corrections is a non-commutative theory with a special kind of non-commutative algebra. We have shown that the non-commutativity can be understood in the Fedosov's deformation quantization on the disk.

## Acknowledgement

The author is very grateful to G. Mandal and S. Wadia for bringing his attention to this problem along with very useful discussions and correspondence. Thanks also to A. Dabholkar, S. Mukhi, and N. V. Suryanarayana for comments and also S. Trivedi for discussion.

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